3. EQUATIONS FOR NUMERICAL EVALUATION

As a first step in obtaining a computation formula for (29), the factor, exp(2f), in the integrand is expanded in series:

$$e^{2f} = \sum_{m=0}^{\infty} (2^m/m!)f^m \quad . \tag{34}$$

Equation (29) may then be written as

$$A = (1/2^N)\ C_N e^{\sigma_N} \sum_{m=0}^{\infty} I_m \quad , \tag{35}$$

$$I_m = (2^m/m!)(2/\sqrt{\pi})^N \int_{\beta_1}^{\infty} \cdots \int_{\beta_N}^{\infty} f^m\, e^{-(x_1{}^2 + \ldots + x_N{}^2)}\, dx_1 \ldots dx_N, \tag{36}$$

$$f = \sum_{j=1}^{N-1} \alpha_j(x_j - \beta_j)(x_{j+1} - \beta_{j+1}) \ , N \geq 2 \quad . \tag{37}$$

For notational convenience, we now define

$$\alpha_N \equiv 1,\ m_o \equiv m,\ m_k \equiv 0 \text{ when } k \geq N - 1 \quad . \tag{38}$$

Then for $N \geq 3$, it can be shown that the expansion of f^m appearing in (36) is expressible as

$$f^m = m! \sum_{m_1=0}^{m} \cdots \sum_{m_{N-2}=0}^{m_{n-3}} \prod_{i=1}^{N} \left[\alpha_i^{\,m_{i-1}-m_i}\ (x_i - \beta_i)^{n_i}/(m_{i-1} - m_i)! \right], \tag{39}$$

where

$$n_i = \begin{cases} (m_0 - m_1) & , \quad i = 1 & (40a) \\ (m_{i-2} - m_i) & , \quad 2 \leq i \leq N - 1 & (40b) \\ (m_{N-2} - m_{N-1}) & , \quad i = N & (40c) \end{cases}$$

The next step is to introduce the functions known as repeated integrals of the error function, $i^n\text{erfc}(z)$, defined by the relationship (Abramowitz and Stegun, 1964, p. 299)

$$(2/\sqrt{\pi}) \int_{\beta}^{\infty} (x - \beta)^n e^{-x^2} dx = n! \iota^n \mathrm{erfc}(\beta) \equiv n! I(n, \beta) \quad . \tag{41}$$

In the equations that follow, the inconvenient notation, $\iota^n \mathrm{erfc}(z)$, has been replaced by the symbol, $I(n, z)$, as indicated in (41).

Now with the use of (39) and (41), (36) can be written as

$$I_m = 2^m \sum_{m_1=0}^{m} \cdot \cdot \cdot \sum_{m_{N-2}=0}^{m_{N-3}} \prod_{i=1}^{N} \left\{ \frac{(m_{i-1} - m_{i+1})!}{(m_i - m_{i+1})!} \right\} \alpha_i^{m_{i-1}-m_i} \quad I(n_i, \beta_i), \; N \geq 3 \tag{42a}$$

where n_i is given by (40) and the definitions of (38) are assumed. Notice that for $N = 2$, f as defined in (37) contains only one term and I_m in this case is simply

$$I_m = 2^m \; m! \; \alpha_1^{\;m} \; I(m, \beta_1) \; I(m, \beta_2), \; N = 2 \quad . \tag{42b}$$

Thus, the equation for the multiple knife-edge attenuation function, $A(N \geq 2)$, is given by (35), where I_m is computed from (42a) or (42b).

It would appear, at first, that the series in (35) might be rather restricted in its range of application because of convergence difficulties. In fact, it provides a suitable means of computation over a wide range of the input parameters, α_i and β_i. This arises from two circumstances: (1) α_i always lies between zero and unity, and (2) the magnitude of the function, $I(n, \beta)$, becomes very small as n increases and as long as β is not too large a negative number. Fortunately, a negative β occurs only when the knife-edge with which it is associated becomes of less and less significance in the overall diffraction problem. Eventually, the attenuation is computed as if that particular knife-edge were absent altogether. It turns out, as will be shown in the example computations, that the series in (35) is suitable for β's just negative enough to approach the correct attenuation value, i.e., the value obtained with one less knife-edge.

The repeated integrals of the error function, $I(n, \beta)$, require different computational algorithms for different ranges of the variables, n and β, in order to achieve sufficient numerical accuracy. The range limits of n and β will vary somewhat for different computers because of significant figure and storage capacity considerations. The algorithms used in the present study are described in the following discussion.

For small z and n not too large, the power series expansion of $I(n, z)$ was found to give satisfactory results (Abramowitz and Stegun, 1964, p. 299).

For $|z| < 0.8$, $n < 10$:

$$I(n, z) = \sum_{k=0}^{\infty} (-1)^k z^k / (2^{n-k} k!\ \Gamma\{1 + (n - k)/2\})$$

$$= \sum_{r=0}^{\infty} T_r^{e}(n, z) - \sum_{r=0}^{\infty} T_r^{o}(n, z) \quad , \tag{43}$$

where $\Gamma(X)$ denotes the usual Gamma Function and

$$T_r^{e}(n, z) = \left\{ \frac{(2 + n - 2r)z^2}{r(2r - 1)} \right\} T_{r-1}^{e}(n, z) \quad , \tag{44a}$$

$$T_r^{o}(n, z) = \left\{ \frac{(1 + n - 2r)z^2}{r(2r + 1)} \right\} T_{r-1}^{o}(n, z) \quad , \tag{44b}$$

$$T_o^{e}(n, z) = 1/2^n\, \Gamma\left(\frac{2 + n}{2}\right) \quad , \quad T_o^{o}(n, z) = 2z/2^n\, \Gamma\left(\frac{1 + n}{2}\right) \quad . \tag{44c}$$

For larger n, an equation derived by Miller (1955, p. 66) was used. For $|z| < 0.8$, $n \geq 10$:

$$I(n, z) = \left[e^{-Z^2} / \ 2^n\, \Gamma\left(\frac{2 + n}{2}\right) \right] e^{V(Z)} \quad , \quad Z \equiv z/\sqrt{2} \quad , \tag{45}$$

$$\text{where } V(Z) = -2\sqrt{n + 1/2}\ \ Z + \sum_{k=1}^{9} g_k / (2\sqrt{n + 1/2})^k \quad , \tag{46}$$

and

$$\begin{aligned} g_1 &= -(2/3)Z^3 \quad , \quad g_2 = -Z^2 \quad , \\ g_3 &= -Z + (2/5)Z^5 \quad , \quad g_4 = 2Z^4 \\ g_5 &= (16/3)Z^3 - (4/7)Z^7 \quad , \quad g_6 = 9Z^2 - (16/3)Z^6 \quad , \\ g_7 &= (19/2)Z - 26Z^5 + (10/9)Z^9 \quad , \\ g_8 &= -84Z^4 + 16Z^8 \quad , \\ g_9 &= -(575/3)Z^3 + 120Z^7 - (28/11)Z^{11} \quad . \end{aligned} \tag{47}$$

If the exponent range and word size of the computer variables are large enough, a method of computing I(n, z) has been developed by Gautschi (1961) based on a technique originated by J. C. P. Miller.

For $|z| \geq 0.8$, Re $z \geq 0$:

$$I(n, z) = (2/\sqrt{\pi})\, e^{-z^2} \{w_n(z)/w_{-1}(z)\}, \; n = 0, 1, \ldots, M \quad , \tag{48}$$

where the auxiliary functions, w, are recursively defined by

$$w_\mu(z) = 2\{(\mu + 2)\, w_{\mu+2}(z) + z w_{\mu+1}(z)\}, \; \mu = \nu, \nu-1, \ldots, 1, 0, -1, \tag{49a}$$

$$w_{\nu+2}(z) \equiv 0, \; w_{\nu+1}(z) \equiv \alpha \quad , \tag{49b}$$

and α is some (arbitrary) small, positive constant.

Gautschi has provided a means of determining how large ν must be as a function of M in (48) in order to obtain a given accuracy. Thus, if we wish to have

$$|(I_{approx} - I_{true})/I_{true}| \leq 10^{-p} \quad , \tag{50}$$

then $\nu \geq \left\{\sqrt{M} + (\ln 10)(p + \log 2)/(2^{3/2}|z|)\right\}^2 \equiv \left\{\sqrt{M} + c/|z|\right\}^2$. (51)

The value of c used in the computer program described later on in Section 4 is $c = 6.758$, which corresponds to $p = 8$. It is obvious, of course, that one of the factors determining what value of p is chosen is the number of significant figures available in the computer that is used.

Finally, for larger negative z, the equation used to compute I(n, z) is obtainable from relationships given by Abramowitz and Stegun (1964, pp. 300 and 775).

For $|z| \geq 0.8$, Re $z < 0$:

$$I(n, z) = 2A_n(z) - (-1)^n\, I(n, -z) \quad , \tag{52}$$

where $$A_n(z) = \sum_{k=0}^{[n/2]} \frac{z^{n-2k}}{4^k k!(n - 2k)!} \quad ; \tag{53}$$

the symbol, [x], in (53) denotes the largest integer $\leq$ x.

One further modification of the computation formula for the function, I_m, given by (42a) was made in order to shorten the computation time. When the number of knife-edges, N, is greater than 4 or 5 and the parameter, m, becomes large, many terms are required in the computation of I_m. If (42a) were programmed as it stands, a number of sub-calculations entering into I_m would be completely recalculated when computing I_{m+1}. If enough storage locations are available, these sub-calculations can be stored for later use, and computation time can be considerably reduced at the expense of increased storage requirements.

Although the algebra is tedious and will not be detailed here, it can be shown that I_m is expressible in the following form. First, we define the function

$$C(N-1, m_{N-2}, m_{N-3}) = (m_{N-3})!\ \alpha_{N-1}^{m_{N-2}}\ I(m_{N-3}, \beta_{N-1})\ I(m_{N-2}, \beta_N) \quad . \tag{54}$$

Then, with the notation

$$i = m_{N-L},\ j = m_{N-L-1},\ k = m_{N-L-2} \quad , \tag{55a}$$

$$2 \leq L \leq N-2 \text{ , for } N \geq 4 \quad , \tag{55b}$$

and the recursive relationship

$$C(N-L, j, k) = \sum_{i=0}^{j} \left\{ \frac{(k-i)!}{(j-i)!} \right\} \alpha_{N-L}^{j-i}\ I(k-i, \beta_{N-L})\ C(N-L+1, i, j) \quad , \tag{56}$$

it can be shown that I_m is given by

$$I_m = 2^m \sum_{m_1=0}^{m_0} \alpha_1^{m_0-m_1}\ I(m_0 - m_1, \beta_1)\ C(2, m_1, m_0) \quad , \tag{57}$$

where, as before, $m_0 \equiv m$.

4. EXAMPLE CALCULATIONS

A computer program has been written to calculate multiple knife-edge attenuation over paths consisting of up to a maximum of 10 knife-edges. The input for a particular propagation path of N knife-edges ($1 \leq N \leq 10$) requires the radio frequency, f (in MHz), the $N + 1$ separation distances, r_n (in kilometers), $n = 1, \ldots, N + 1$, and the $N + 2$ antenna and knife-edge heights, h_n (in kilometers above some reference plane), $n = 0, \ldots, N + 1$. The symbols r_1 and r_{N+1} denote the distances from one antenna to the first knife-edge and from the Nth knife-edge to the other antenna, respectively; h_o and h_{N+1} denote the heights of the antennas. One restriction on the separation distances, arising from the derivation of the attenuation function, is that kr_n always should be much greater than unity.

As can be seen from (23), the attenuation A is a function of the angles, θ_n, appearing in the definition of β_n. These angles are approximately related to the heights and distances, h_n and r_n, by

$$\theta_n \simeq \frac{h_n - h_{n-1}}{r_n} + \frac{h_n - h_{n+1}}{r_{n+1}} \;, \; n = 1, \ldots, N \quad ; \tag{58}$$

θ_n is in radians and may be either positive or negative. The approximation in (58) is suitable for small θ such that $\tan \theta \simeq \theta$.

The actual calculation of A from the equation in (35) must, of course, be restricted to a finite number of terms. In order to achieve sufficient accuracy more terms are needed as the number of knife-edges is increased. However, for $N > 3$ no previous results are available which can be used to check the answers obtained from (35). Fortunately, an exact expression can be derived for multiple knife-edge attenuation as given in the integral form of (29) for the special case of equal separation distances and θ_n (or β_n) equal to zero. Thus, for

$$r_1 = r_2 = \cdot \cdot \cdot \cdot = r_{N+1} = \text{constant} \quad , \tag{59a}$$

$$h_o = h_1 = \cdot \cdot \cdot \cdot = h_{N+1} = \text{constant} \quad , \tag{59b}$$

we have, from (58), (23), and (24),

$$\beta_n = 0 \; (n = 1, \ldots, N), \; \alpha_n = 1/2 \; (n = 1, \ldots, N - 1) \quad . \tag{60}$$

Then it can be shown that the multiple knife-edge attenuation for N knife-edges as given by (29) is

$$A(N) = \frac{1}{N + 1} \quad . \tag{61}$$

Now, if for practical programming purposes, (35) is approximated by

$$A = (1/2^N)\ C_N e^{\sigma_N} \sum_{m=0}^{M} I_m \ , \tag{62}$$

equation (61) can be used to estimate the value of M necessary to achieve a given accuracy. Considerations of computer storage limitations and exponent ranges further limit the choice of M and, after some experimentation, a maximum value of M = 160 was selected for the present program on this particular computer. Comparisons of results obtained from (62) with the exact value as given by (61) are shown in Table 1.

Table 1. Comparisons of Multiple Knife-Edge Attenuation, A, as Obtained from (61) and (62) for Input Parameters as in (59)

N	M	Exact A from (61)	A from (62)	Time (s)
5	90	$0.1\dot{6}$	0.166667	1.2
6	100	0.142857	0.142855	2.4
7	160	0.125	0.12499975	12.1
8	160	$0.\dot{1}$	0.111107	15.4
9	160	0.1	0.0999674	18.8
10	160	$0.\dot{0}\dot{9}$	0.0907650	21.2

The column headed "Time" shows the amount of computer time (in seconds) used in obtaining the attenuations of column 4.

As stated previously, the number of terms in (62) necessary to achieve a given accuracy increases as the number of knife-edges increases. Table 1 shows that for 10 knife-edges and using 161 terms, the result from (62) is barely good to three figures. In terms of decibels the approximate result differs by 0.014 dB from the exact value, and this is sufficiently accurate for measurement purposes. The amount of computer time used drops dramatically as the number of knife-edges is decreased (and, consequently, M may be chosen smaller). For example, for six knife-edges and with M = 100, (62) gives A = 0.142855, as against the exact value: A = 0.142857. The computation time in this case is 2.4 seconds.

The above discussion is useful in checking the validity of (62) when $\beta_m \geq 0$. The series continues to provide valid results for negative β as long as $|\beta|$ is not too large. In complete analogy with the series expansion of exp(-x), the series is valid but impractical for computation because of the loss of figures in the addition and subtraction of large numbers.

In the knife-edge diffraction problem a knife-edge has a significant effect on the signal only when it obstructs or is near the ray path. As it drops lower and lower below the ray, its effect diminishes. Numerical studies of (62) for negative β's show that the series gives satisfactory estimates of the magnitude of the attenuation to the point where the knife-edge (or knife-edges) can be neglected. However, the phase of the attenuation near this changeover point should not be trusted because of the fact that θ becomes (negatively) large and the angle approximation in (58) is less reliable.

Investigations to ascertain suitable values for the minimum β have shown that the values depend on the number of knife-edges in the path. Table 2 shows the minimum β, BR_{min}, used with each N and also gives comparisons of attenuation when the knife-edge height is at the "changeover" value. The input heights and distances are such that when a particular knife-edge height, h_n, is just low enough to be considered insignificant, the remaining heights and distances give θ's equal to zero and α's equal to 0.5; thus, the attenuation is given by (61) for the reduced number of significant knife-edges. When the knife-edge, h_n, is just above the "changeover" height, all the input heights are significant, yet the attenuation still should be approximately equal to that for the reduced case.

It should be realized that (62) was used to calculate both A(N) and $A(N_{eff})$ in Table 2. For instance, in the case of five knife-edges, h_2 and h_4 were input with values just above the changeover height. Thus, the program considered all five input heights, h_1 through h_5, as significant and calculated the value $|A(5)| = 0.247253$. Next, the program was run with h_2 and h_4 just below the changeover height. In this case the program considered only the three heights h_1, h_3, and h_5 as significant and calculated the value $|A(3)| = 0.250000$. A similar procedure was used in each of the other entries.

The example with nine knife-edges shows the greatest discrepancy between the two attenuation values in decibels, i.e., the magnitude of the difference is 0.22 dB. It would be difficult to state an analytical error estimate for (62) because of the multiple summation form of the I_m functions. A numerical estimate for any particular set of input parameters can be obtained by comparing M and M - 1 terms.

Table 2. Comparisons of Multiple Knife-Edge Attenuation at the Changeover Value, $\text{Re}\beta = BR_{min}$

N	BR_{min}	$\|A(N)\|$	$\|A(N_{eff})\|$	N_{eff}
2	- 3.0	0.494791	0.500000	1
3	- 3.0	0.333172	0.333333	2
4	- 1.5	0.248615	0.250000	3
5	- 1.5	0.247253	0.250000	3
6	- 1.2	0.200630	0.200000	4
7	- 1.2	0.201019	0.200000	4
8	- 1.0	0.169766	0.166667	5
9	- 1.0	0.170914	0.166667	5
10	- 1.0	0.143444	0.142857	6

Many combinations of knife-edge heights near their changeover values were tested other than the ones shown in Table 2. The largest differences occur for the cases of 8, 9, or 10 knife-edges. In all the tests made, the greatest difference was found for a path with $N = 9$ in which one of the separation distances was chosen to be $r = 0.01$ km, a value that might be considered the minimum allowable. The dB difference of the answers for $A(9)$ and the reduced case of $A(8)$ was 0.85 dB. It is believed that the present program will always give estimates of attenuation good to within 1 dB of the theoretical value.

One additional, but restricted, means of verifying (62) is through comparison with the results of the double and triple knife-edge computer programs previously mentioned. These programs, written a number of years ago but never published, use different series expansions for various ranges of the input parameters and are said to give attenuation values accurate to eight significant figures. Comparisons of these two programs with the present program based on (62) always gave answers in agreement to six or more figures as long as the knife-edge heights were greater than the "changeover" value and the separation distances were greater than 0.01 km. When comparisons were made using heights near or below the changeover, the dB difference of the answers never exceeded 0.2 dB. In fact in many cases it was found that the series in (62) could be used with values of β much less than -3.0, resulting in four- and five-figure agreement in the answers. In other cases the addition and subtraction of the series terms resulted in the loss of too many figures, and it was finally decided that a minimum β of -3.0 was best suited for all cases of double and triple knife-edge diffraction.

5. SUMMARY

A multiple knife-edge diffraction theory has been developed starting from Furutsu's generalized residue series formulation for the propagation of electromagnetic waves over a sequence of smooth, rounded obstacles. The resulting expression, in the form of a multiple integral [see equation (29)], is transformed into the series (35) through the use of repeated integrals of the error function. The terms of the series, I_m, are defined by (42).

A computer program has been written to calculate the magnitude of the attenuation relative to free space for propagation over paths containing N knife-edges ($N \leq 10$). The program uses equation (62) with I_m given by (57), C_N by (27), and σ_N by (31). The basic parameters β_i and α_i are defined in (23) and (24), respectively.

Comparison of the program with previously written double and triple knife-edge programs shows six significant figure agreement as long as all $\beta_i \geq -3.0$. In all cases tested the dB difference in answers was always less than 0.2 dB.

Since no previous results exist when the number of knife edges is greater than three, partial verification of the program was made in two ways.

1. Answers were compared with a closed form expression valid when all knife-edges are evenly spaced and at equal heights such that all $\theta_i = 0$. The largest dB difference occurred for the case of N = 10, this difference being 0.014 dB (see Table 1).

2. Answers were compared with sample paths in which some of the knife-edges were lowered to the point where the attenuation would be that expected for the path with a reduced number of knife-edges (see Table 2). In general the test case answers agreed to within 0.4 dB. Some paths containing minimum separation distances of 0.01 km gave larger discrepancies, but in all cases the answers agreed to within 1 dB.

The multiple knife-edge attenuation function described in this paper should serve as a useful means of estimating propagation loss for microwave frequency propagation paths over irregular terrain. Even at lower frequencies the model is applicable if the terrain features can be characterized as knife-edges.